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Galerkin approximation for elliptic PDEs on spheres

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Abstract

We discuss a Galerkin approximation scheme for the elliptic partial differential equation $-\Delta u + \omega^2 u = f$ on $S^n \subset \mathbb{R}^{n+1}$. Here Δ is the Laplace–Beltrami operator on S^n, ω is a non-zero constant and f belongs to $C^{2k-2}(S^n)$, where $k \geqslant n/4+1$, k is an integer. The shifts of a spherical basis function ϕ with $\phi \in H^\tau(S^n)$ and $\tau > 2k \geqslant n/2+2$ are used to construct an approximate solution. An $H^1(S^n)$-error estimate is derived under the assumption that the exact solution u belongs to $C^{2k}(S^n)$.

Keywords: Spherical basis function; Galerkin method

1. Introduction

The theory of interpolation and approximation of solutions to differential and integral equations on spheres has attracted considerable interest in recent years; it has also been applied fruitfully in fields such as physical geodesy, potential theory, oceanography, and meteorology [7,9,15]. As more satellites are being launched into space, the acquisition of global data is becoming more important, and there is a growing demand for the processing and mathematical modelling of such data.

Differential or, more generally, pseudodifferential equations arise in many areas of earth sciences. Pseudodifferential operators of order t on the sphere are operators with

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eigenvalues $\Lambda(\ell),\ \ell = 0, 1, \ldots,$ which are asymptotic to $(\ell + 1/2)^t$. A detailed discussion on pseudodifferential operators and their applications can be found in [3,9,11,31].

Given a pseudodifferential operator Ψ and a continuous function f which is defined on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, we shall discuss the approximation of solutions of the equation

$$\Psi u = f \text{ on } S^n.$$

The approximate solution will be constructed as a linear combination of spherical basis functions which are derived from zonal kernels $\Phi : S^n \times S^n \to \mathbb{R}$ of the form

$$\Phi(x, y) = \phi(x \cdot y), \qquad x, y \in S^n,$$

where ϕ is a univariate function defined on $[-1, 1]$, and $x \cdot y$ is the Euclidean dot product of the position vectors of the points $x, y \in S^n$. For a fixed x the value of $\Phi(x, y)$ depends only on the geodesic distance from x to y, so the function $\Phi(x, \cdot)$ is radially symmetric with respect to the point x, and is called a spherical basis function (SBF). A linear combination of SBFs is called *spherical spline* as in [10].

In [10], a collocation method based on SBFs is used to approximate the solutions of a class of pseudodifferential equations on S^n. The collocation method requires the approximate solution to satisfy the differential equations at a given set of points on the unit sphere. In [15], various Sobolev error bounds for solving pseudodifferential equations on spheres are given for the collocation method using spherical splines based on the smoothness of the kernel $\Phi(x, y)$. However, the results in [15] have a disadvantage that the function f is required to be in a subspace of the *native space* induced by Φ (see Section 2).

In this paper, we shall use the Galerkin method, with the approximate solution being spanned by spherical basis functions. Together with recent results in the theory of interpolation of continuous functions by spherical basis functions (see [5,18,20]), we can relax the smoothness of f and let f escape to $C^{2k}(S^n)$ for some $k \geqslant 1$, which is larger than the native space. For a domain $\Omega \subset \mathbb{R}^{n+1}$, the idea of using Galerkin method for solving elliptic partial differential equations in which the approximate solution is constructed from a linear combination of shifts of a radial basis function on a scattered set of points has been introduced in [34].

We shall restrict Ψ to a class of elliptic differential operators of the form $-\Delta + \omega^2$, where Δ is the Laplace–Beltrami operator on the sphere and $\omega \neq 0$. This form of operators arise frequently in the time discretization using the Euler method of the heat or the wave equation on spheres. With a slight modification, our approach could be used to analyze an arbitrary invertible pseudodifferential operator of order 2, such as the operator of second-order radial derivative at the earth's mean radius R, which has eigenvalues $(\ell + 1)(\ell + 2)/R^2$ and is of basic importance in satellite gradiometry (cf., e.g., [10,24,28]). The other classes of pseudodifferential operators such as the Stoke integral operator, the integral of the single-layer potential, the double-layer potential, etc. are deferred for future research.

We aim to make use of recent results in [20] to derive error estimates for the Galerkin approximation on S^n of the elliptic partial differential equation

$$-\Delta u(x) + \omega^2 u(x) = f(x), \qquad x \in S^n,$$

where ω is a non-zero real constant, Δ is the Laplace–Beltrami operator on S^n, and $f \in C^{2k}(S^n)$ for some $k \geqslant 1$.

The finite-dimensional subspace used to approximate the solution of the PDE will be the space of shifts of a spherical basis function (see Section 2). Such spaces are used extensively in the interpolation problem on spheres [5,18,19,20]. Assuming that the exact solution u is in $C^{2k}(S^n)$, the main result of this paper (Theorem 5.1) is a Sobolev type error estimate for $u-u_h$, where u_h is the finite element approximation of u, constructed using SBFs satisfying certain regularity conditions.

The paper is organized as follows: Section 2 gives the necessary background on spherical harmonics and the Laplace–Beltrami operator. In Section 3 we outline the weak formulation of the PDE on the unit sphere, and prove a version of Cea's lemma on the unit sphere. In Section 4 we present the error estimates in the supremum norm as well as the Sobolev norm in $H^1(S^n)$. The last section describes some numerical experiments involving data points on S^2.

2. Preliminaries

2.1. Spherical harmonics

A detailed discussion on spherical harmonics can be found in [16]. In brief, spherical harmonics are restrictions to the unit sphere S^n of polynomials $Y(x)$ which satisfy $\Delta_x Y(x) = 0$, where Δ_x is the Laplacian operator in $\mathbb{R}^{n+1}$. The space of all spherical harmonics of degree ℓ on S^n, denoted by V_ℓ, has an orthonormal basis

$$\{Y_{\ell k} : k = 1, \ldots, N(n,\ell)\},$$

where

$$N(n,0) = 1 \text{ and } N(n,\ell) = \frac{(2\ell+n-1)\Gamma(\ell+n-1)}{\Gamma(\ell+1)\Gamma(n)} \text{ for } \ell \geqslant 1.$$

The eigenfunctions of the Laplace–Beltrami operator are the spherical harmonics Y_ℓ; more precisely,

$$-\Delta Y_\ell = \lambda_\ell Y_\ell, \quad \lambda_\ell = \ell(\ell+n-1).$$

The space of spherical harmonics of order L or less will be denoted by $\mathcal{V}_L := \sum_{\ell=0}^{L} V_\ell$; it has dimension $N(n+1,\ell)$. Every function $f \in L^2(S^n)$ can be expanded in terms of spherical harmonics:

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{f}_{\ell k} Y_{\ell k}, \qquad \widehat{f}_{\ell k} = \int_{S^n} f \overline{Y_{\ell k}}\, dS,$$

where dS is the surface measure of the unit sphere. The $L^2(S^n)$-norm of f, given by the familiar formula

$$\|f\|_2 = \left(\int_{S^n} |f|^2 \, dS \right)^{1/2},$$

can also be expressed, via Parseval's identity, as follows:

$$\|f\|_2 = \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} |\widehat{f}_{\ell k}|^2 \right)^{1/2}.$$

The Sobolev space $H^s := H^s(S^n)$ on the sphere is defined as follows:

$$H^s := \{ f \in L^2(S^n) : \|f\|_{H^s}^2 := \sum_{\ell=0}^{\infty} (1+\lambda_\ell)^s \sum_{k=1}^{N(n,\ell)} |\widehat{f}_{\ell k}|^2 < \infty \}.$$

2.2. Interpolation of scattered data on S^n

Let $X = \{x_1, \ldots, x_m\}$ be a finite set of distinct points on S^n. The density of the set X is measured by the mesh norm

$$h_X = \sup_{y \in X} \operatorname{dist}(y, X),$$

where $\operatorname{dist}(y, X) = \inf_{x \in X} \theta(y, x)$. Here θ is the geodesic distance on S^n which is defined as $\theta(x, y) = \cos^{-1}(x \cdot y)$, where x and y are represented as two unit vectors in $\mathbb{R}^{n+1}$. The separation radius of the set X is defined via

$$q_X = \tfrac{1}{2} \min_{j \neq k} \theta(x_j, x_k).$$

It is easy to see that $h_X \geqslant q_X$; equality can hold only for a uniform distribution of points on the circle S^1. The *mesh ratio* $\rho_X := h_X / q_X \geqslant 1$ provides a measure of how uniformly points in X are distributed on S^n. If there is a constant C independent of X such that $\rho_X \leqslant C$ then the set X is called *quasi-uniform*.

Bizonal functions on S^n are functions that can be represented as $\phi(x \cdot y)$ for all $x, y \in S^n$, where $\phi(t)$ is a continuous function on $[-1, 1]$. We shall be concerned exclusively with bizonal kernels of the type

$$\Phi(x, y) = \phi(x \cdot y) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(n+1; x \cdot y), \quad a_\ell \geqslant 0, \quad \sum_{\ell=0}^{\infty} a_\ell < \infty, \tag{1}$$

where $\{P_\ell(n+1; t)\}_{\ell=0}^{\infty}$ is the sequence of $(n+1)$-dimensional Legendre polynomials. Recall from [16] that

$$\int_{-1}^{1} P_\ell(n+1; t) P_k(n+1; t)(1-t^2)^{(n-2)/2}\, dt = 0 \text{ for } \ell \neq k$$

and

$$\int_{-1}^{1} [P_\ell(n+1; t)]^2 (1-t^2)^{(n-2)/2}\, dt = \frac{|S^n|}{|S^{n-1}| N(n, \ell)},$$

where $|S^n|$ is the surface area of S^n and $|S^{n-1}|$ is the surface area of S^{n-1}.

Thanks to the seminal work of Schoenberg [27], we know that such a Φ is positive definite on S^n, that is, the matrix $A := [\Phi(x_i, x_j)]_{i,j=1}^m$ is positive semidefinite for every set of distinct points $\{x_1, \ldots, x_m\}$ on S^n and every positive integer m. When the coefficients a_ℓ are positive for every ℓ, we say that Φ is strictly positive definite. In this case the matrix A becomes positive definite, hence invertible, for every set of distinct points $\{x_1, \ldots, x_m\}$ on S^n and every m (see [35]).

In particular, the following interpolation problem admits a unique solution: given a strictly positive definite Φ, a continuous function f on S^n, a positive integer m, and a set of distinct points $X = \{x_1, \ldots, x_m\}$ on S^n, there exists a unique sequence of numbers $\{c_j\}_{j=1}^m$ such that the function

$$f_X(x) = \sum_{j=1}^{m} c_j \Phi(x, x_j) \tag{2}$$

satisfies the interpolatory conditions

$$f_X(x_k) = f(x_k), \quad 1 \leqslant k \leqslant m.$$

Using the addition theorem for spherical harmonics (see, for example, [17, p. 18]), we can write

$$\Phi(x, y) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell k}(x) \overline{Y_{\ell k}(y)}, \text{ where } \widehat{\phi}(\ell) = \frac{|S^n|}{N(n, \ell)} a_\ell. \tag{3}$$

Throughout the paper, we make a further assumption that $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\tau}$ for some $\tau > n/2 + 2$, i.e. there exist positive constants β_1, β_2 such that

$$\beta_1 (1 + \lambda_\ell)^{-\tau} \leqslant \widehat{\phi}(\ell) \leqslant \beta_2 (1 + \lambda_\ell)^{-\tau}, \quad \ell \geqslant 0. \tag{4}$$

The *native space* induced by Φ is defined to be the closure of the set

$$N_\Phi := \left\{ f \in \mathcal{D}'(S^n) : \|f\|_\Phi^2 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} |\widehat{f}_{\ell k}|^2 / \widehat{\phi}(\ell) < \infty \right\},$$

where $\mathcal{D}'(S^n)$ denotes the set of all tempered distributions defined on S^n.

In what follows, the supremum norm in $C(S^n)$ will be denoted by $\|\cdot\|$; for later use, we also introduce the following norm in $C^{2k}(S^n)$:

$$\|f\|_{2k} := \max\{\|f\|, \|\Delta^k f\|\}, \qquad f \in C^{2k}(S^n).$$

The main result in [20, Theorem 3.2] asserts the following:

Theorem 2.1 (Narcowich and Ward [20]). *Let Φ be an SBF of the form* (3), *with $\widehat{\phi}(\ell)$ satisfying condition* (4) *for $\tau > 2k \geqslant n/2$. If $X = \{x_1, \ldots, x_m\}$ is a set of distinct points on S^n, $f \in C^{2k}(S^n)$, and f_X is defined as in* (2), *then*

$$\|f - f_X\| \leqslant C \rho_X^{\tau-2k} h_X^{2k-n/2} \|f\|_{2k},$$

where C is independent of f and X, and h_X and ρ_X are the mesh norm and mesh ratio for the set X, respectively.

Remark. If the mesh ratio ρ_X is bounded by a constant, then the error estimates only depend on the mesh norm, i.e.,

$$\|f - f_X\| = \mathcal{O}(h_X^{2k-n/2}).$$

2.3. Positive definite kernels and the power function

A conjugate symmetric, complex-valued kernel $\Phi \in C(S^n \times S^n) \cap H^{2s}(S^n \times S^n)$ is said to be *positive definite* if for every finite subset $X = \{x_1, \ldots, x_m\} \subset S^n$ of m distinct points, the matrix A with entries $A_{i,j} = \Phi(x_i, x_j)$ is positive semidefinite. In terms of distributions, the positive definiteness of Φ is equivalent to the following [5, Theorem 2.1]: for every non-zero distribution w in the dual Sobolev space $H^{-s}(S^n)$,

$$(\overline{w} \otimes w, \Phi) := \int_{S^n} w(x) \left(\int_{S^n} w(y)\Phi(x, y)\, dS(y) \right) dS(x) \geqslant 0.$$

If $(\overline{w} \otimes w, \Phi) > 0$ for every $w \neq 0$, we will call Φ *strictly positive definite*. The kernel Φ is positive definite (or strictly positive definite) if and only if all the coefficients a_ℓ in the Legendre polynomial expansion (1) are non-negative (or positive) [18]. We define

$$\Phi * w(x) := (\delta_x \otimes w, \Phi), \quad x \in S^n,$$

where δ_x is the Dirac point evaluation functional. Let $\mathcal{U}$ be a finite-dimensional subspace of functions in $C^k(S^n)$, and let $\mathcal{U}^\perp$ be a space of all distributions over $C^k(S^n)$ such that $(\overline{w}, p) = 0$ for all $p \in \mathcal{U}$. Given a strictly positive definite kernel Φ, we can define an inner product on $\mathcal{U}^\perp$ and the correspondent norm as

$$[v, w]_\Phi := (\overline{v} \otimes w, \Phi), \quad v, w \in \mathcal{U}^\perp, \text{ and } [\![v]\!]_\Phi := \sqrt{[v, v]_\Phi}, \quad v \in \mathcal{U}^\perp.$$

The interpolation problem can be put into a distributional framework in the following way. Let $W = \{w_1, \ldots, w_m\}$ be a linearly independent set of distributions defined on $C^k(S^n)$, and let f be a function in $C^k(S^n)$. Given the data $d_j = (\overline{w}_j, f)$, $j = 1, \ldots, m$, we seek to find $w \in \text{span}\{W\} \cap \mathcal{U}^\perp$ and $p \in \mathcal{U}$ such that $f_X = \Phi * w + p$ satisfies $(\overline{w}_j, f_X) = d_j$ for every $1 \leqslant j \leqslant m$, and if $f \in \mathcal{U}$, then $f_X = p = f$. The latter requirement that the interpolation process reproduces $\mathcal{U}$ implies that the set $\overline{W}|_\mathcal{U} = \{\overline{w}_1|_\mathcal{U}, \ldots, \overline{w}_m|_\mathcal{U}\}$ spans $\mathcal{U}^*$, the dual of $\mathcal{U}$.

Suppose that the function f generating the data has the form $f = \Phi * v + q$, with $q \in \mathcal{U}$ and $v \in \mathcal{U}^\perp$. Let η be a distribution defined on functions in $C^k(S^n)$, for example $\eta = \delta_x$. In order to estimate the error $f - f_X$, we need to estimate $|(\overline{\delta_x}, f - f_X)|$ for every value of x. For a general η, in order to estimate $|(\overline{\eta}, f - f_X)|$, we observe that, by construction, $(\overline{w}_j, f - f_X) = 0$ for $j = 1, \ldots, m$; and so if we can find c_j's such that $\eta - \sum_{j=1}^m c_j w_j$

is in $\mathcal{U}^\perp$, then

$$
\begin{aligned}
(\overline{\eta}, f - f_X) &= \left(\overline{\eta - \sum_j c_j w_j}, \Phi * (v - w) + q - p \right) \\
&= \left(\overline{\eta - \sum_j c_j w_j}, \Phi * (v - w) \right) \\
&= \left[v - w, \eta - \sum_j c_j w_j \right]_\Phi .
\end{aligned} \tag{5}
$$

If we set $\eta = w \in \mathcal{U}^\perp \cap \operatorname{span}\{W\}$ in (5) then the left-hand side of (5) is 0 and the right-hand side is $[v - w, w]_\Phi = 0$, since we can take all c_j's to be 0. It then follows that $[\![v]\!]_\Phi^2 = [\![v - w]\!]_\Phi^2 + [\![w]\!]_\Phi^2$, which yields

$$
[\![w]\!]_\Phi < [\![v]\!]_\Phi \text{ and } [\![v - w]\!]_\Phi < [\![v]\!]_\Phi. \tag{6}
$$

By applying Schwarz's inequality to the right-hand side of (5), and using (6), we obtain

$$
|(\overline{\eta}, f - f_X)| \leqslant [\![v]\!]_\Phi [\![\eta - \sum_j c_j w_j]\!]_\Phi, \text{ where } \sum_j c_j w_j|_{\mathcal{U}} = \eta|_{\mathcal{U}}. \tag{7}
$$

We define the *power function* [26] to be

$$
P^\eta_{\Phi,W} := \min \left\{ [\![\eta - \sum_j c_j w_j]\!]_\Phi : \sum_j c_j w_j|_{\mathcal{U}} = \eta|_{\mathcal{U}} \right\}. \tag{8}
$$

Let $\Phi_{\mathcal{U}} \in \mathcal{U} \otimes \overline{\mathcal{U}}$ be an appropriate conjugate symmetric kernel that approximates Φ. We define

$$
\Delta_0 := |(\bar{\eta} \otimes \eta, \Phi - \Phi_{\mathcal{U}})|,
$$

$$
\Delta_1 := \max_j |(\bar{\eta} \otimes w_j, \Phi - \Phi_{\mathcal{U}})|
$$

and

$$
\Delta_2 := \max_{j,k} |(\bar{w}_k \otimes w_j, \Phi - \Phi_{\mathcal{U}})|.
$$

Theorem 2.2 (Narcowich and Ward [21, Section 3]). *For any set of coefficients satisfying the constraint*

$$
\sum_j c_j w_j|_{\mathcal{U}} = \eta|_{\mathcal{U}},
$$

we have the following bound on the power function:

$$
(P^\eta_{\Phi,W})^2 \leqslant \Delta_0 + 2\|c\|_1 \Delta_1 + \|c\|_1^2 \Delta_2, \text{ where } \|c\|_1 = \sum_j |c_j|. \tag{9}
$$

2.4. Norming sets

In order to bound the term $\|c\|_1$ in the right-hand side of inequality (9), we shall employ *norming sets*, the use of which in the context of scattered data interpolation was initiated in [12].

Let V be a finite-dimensional vector space with norm $\|\cdot\|_V$ and let $Z \subset V^*$ be a finite set of cardinality m. We will say that Z is a norming set for V if the mapping $T : V \to T(V) \subset \mathbb{R}^m$ defined by $T(u) = (z(u))_{z\in Z}$ is injective. The operator T is called the *sampling operator*. The norm of its inverse is given by

$$\|T^{-1}\| = \sup_{v\in V} \left\{ \|v\|_V : \max_{z\in Z} |z(v)| = 1 \right\}.$$

Proposition 2.1 (Mhaskar et al. [14, Proposition 4.1]). *Let Z be a norming set for V with T being the corresponding sampling operator. If $\lambda \in V^*$ with $\|\lambda\|_{V^*} \leqslant A$, then there exist real numbers $\{a_z : z \in Z\}$ depending only on λ such that for every $v \in V$,*

$$\lambda(v) = \sum_{z\in Z} a_z z(v), \text{ and } \sum_{z\in Z} |a_z| \leqslant A\|T^{-1}\|.$$

3. Weak formulation of the PDE

In this section, we set up the weak formulation for a class of elliptic partial differential equations on the unit sphere and prove a version of Cea's lemma for our equation on spheres (see [2] for a version on $\mathbb{R}^n$).

Let ω be a non-zero real constant, and consider the partial differential equation

$$-\Delta u(x) + \omega^2 u(x) = f(x), \quad x \in S^n. \tag{10}$$

The weak formulation of this equation is

$$\left\langle -\Delta u + \omega^2 u, v \right\rangle = \langle f, v \rangle, \quad \forall v \in H^1, \text{ where } \langle u, v\rangle := \int_{S^n} u\overline{v}\, dS.$$

Defining the bilinear form

$$a(u, v) := \left\langle -\Delta u + \omega^2 u, v \right\rangle,$$

we find that the weak formulation becomes

$$a(u, v) = \langle f, v \rangle \quad \forall v \in H^1.$$

Lemma 3.1. *There exist positive constants $C \geqslant 1$ and $\alpha \leqslant 1$ such that*

$$|a(u, v)| \leqslant C\|u\|_{H^1}\|v\|_{H^1} \text{ and } |a(u, u)| \geqslant \alpha\|u\|^2_{H^1}.$$

Proof.

$$a(u, v) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)\widehat{u}_{\ell k}\widehat{v}_{\ell k}$$
$$\leqslant \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)|\widehat{u}_{\ell k}|^2 \right)^{1/2} \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)|\widehat{v}_{\ell k}|^2 \right)^{1/2}$$
$$\leqslant \max\{1, \omega^2\}\|u\|_{H^1}\|v\|_{H^1}.$$

We also have

$$a(u, u) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)|\widehat{u}_{\ell k}|^2 \geqslant \min\{1, \omega^2\} \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (\lambda_\ell + 1)|\widehat{u}_{\ell k}|^2. \qquad \square$$

The preceding lemma shows that the bilinear form $a(u, v)$ is bounded and coercive, so by the Lax–Milgram theorem (cf. [2]), the weak formulation has a unique solution. It is easy to see that $\phi_i(x) := \Phi(x, x_i) = \phi(x \cdot x_i)$ is in H^1 since we require $\tau > n/2 + 2$. We now define a finite dimensional subspace of $H^1(S^n)$:

$$V_X := \operatorname{span}\{\phi_i(x) : i = 1, \ldots, m\}.$$

The Ritz–Galerkin approximation problem is the following:

$$\text{find } u_h \in V_X \text{ such that } a(u_h, \chi) = \langle f, \chi \rangle, \quad \forall \chi \in V_X. \tag{11}$$

The following is a version of Cea's lemma for unit spheres.

Lemma 3.2. *Let $u \in H^1(S^n)$ and $u_h \in V_X$ be the solution of the Ritz–Galerkin approximation problem* (11), *then there exists a constant $C \geqslant 1$ such that*

$$\|u - u_h\|_{H^1} \leqslant C \inf_{v \in V_X} \|u - v\|_{H^1}.$$

Proof. Note that $a(u - u_h, \chi) = 0$ for all $\chi \in V_X$. In particular, $a(u - u_h, v - u_h) = 0$ for any $v \in V_X$. Thus,

$$a(u - u_h, u - u_h) = a(u - u_h, u - v + v - u_h) = a(u - u_h, u - v).$$

By Lemma 3.1, we have

$$\alpha\|u - u_h\|_{H^1}^2 \leqslant a(u - u_h, u - u_h) = a(u - u_h, u - v)$$
$$\leqslant C\|u - u_h\|_{H^1}\|u - v\|_{H^1}.$$

Dividing both sides by $\|u - u_h\|$ and taking infimum over $v \in V_X$, we obtain the required result. $\square$

Lemma 3.3. *For a function $u \in H^1$, the following inequality holds*:

$$\|u\|_{H^1} \leqslant (\|\Delta u\|_2 + \|u\|_2)^{1/2}\|u\|_2^{1/2}.$$

Proof.

$$
\begin{aligned}
\|u\|_{H^1}^2 &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} (\lambda_\ell + 1)|\widehat{u}_{\ell k}|^2 \\
&\leqslant \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \lambda_\ell |\widehat{u}_{\ell k}|^2 + \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} |\widehat{u}_{\ell k}|^2 \\
&\leqslant \left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \lambda_\ell^2 |\widehat{u}_{\ell k}|^2\right)^{1/2} \left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} |\widehat{u}_{\ell k}|^2\right)^{1/2} + \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} |\widehat{u}_{\ell k}|^2 \\
&= \|\Delta u\|_2 \|u\|_2 + \|u\|_2^2. \qquad \square
\end{aligned}
$$

The foregoing lemma enables us to use recent results in [20] to estimate $\|u - u_X\|_{H^1}$, where $u_X \in V_X$ is the interpolant of u on X, i.e $u(x_j) = u_X(x_j)$ for all $x_j \in X$.

4. Estimate for $\|\Delta^s u - \Delta^s u_X\|$

We shall estimate the error in two steps: firstly, u is assumed to be in the native space N_Φ and the error will be bounded by a factor of $\|u\|_\Phi$; secondly, we let u escape to a larger space $C^{2k}(S^n)$ and estimate the error in terms of $\|u\|_{2k}$.

4.1. Estimate in the native space norm

Before proceeding to the main estimate, we need the Markov–Bernstein inequality for spherical polynomials of order L. A proof of this result may be found in [22].

Theorem 4.1. *If $P_L \in \mathcal{V}_L$, then*

$$\|\Delta P_L\| \leqslant D_n L^2 \|P_L\|,$$

where the constant D_n depends only on the dimension of the ambient space.

Remark. It is known that $D_2 = 4$ (see [21]).

Corollary 4.1. *If $P_L \in \mathcal{V}_L$ and s is an integer, then*

$$\|\Delta^s P_L\| \leqslant D_n^s L^{2s} \|P_L\|.$$

Next we need to adapt [21, Theorem 6.4] to the case S^n.

Proposition 4.1. *If the mesh norm of X satisfies $h_X < 1/(2L)$, then for any fixed x there exist numbers $\alpha_j(x), 1 \leqslant j \leqslant m$, such that*

$$\sum_{j=1}^{m} \alpha_j(x) Y(x_j) = \Delta^s Y(x) \ \textit{for all}\ Y \in \mathcal{V}_L,\ \textit{and}\ \sum_{j=1}^{m} |\alpha_j(x)| \leqslant 2 D_n^s L^2.$$

Proof. Let T be the point-sampling operator, namely, $T(Y) = (Y(x_1), \ldots, Y(x_m))$, and let $\lambda(Y) = \Delta^s Y(x)$. The upper bound for $\|\lambda\|$ is given by Corollary 4.1. Moreover, if the mesh norm $h_X < 1/(2L)$ then $\|T^{-1}\| \leqslant 2$ (see [12]). The required result now follows via Proposition 2.1. □

Defining the ordinary differential operator

$$\mathcal{L} := -(1-t^2)^{(2-n)/2}\frac{d}{dt}(1-t^2)^{n/2}\frac{d}{dt} = -(1-t^2)\left(\frac{d}{dt}\right)^2 + nt\frac{d}{dt},$$

we recall from [17, p. 38] that the $(n+1)$-dimensional Legendre polynomials $P_\ell(n+1;t)$ satisfy the differential equation

$$\mathcal{L}P_\ell(n+1;t) = \lambda_\ell P_\ell(n+1;t).$$

The operator $\mathcal{L}$ can be iterated as $\mathcal{L}^{k+1}P = \mathcal{L}(\mathcal{L}^k P)$ for $k \geqslant 1$. We approximate the kernel Φ by the truncated kernel Φ_L:

$$\Phi_L(x,y) = \phi_L(x \cdot y) = \sum_{\ell=0}^{L} a_\ell P_\ell(n+1; x\cdot y),$$

which belongs to the space $\mathcal{V}_L \otimes \overline{\mathcal{V}}_L$.

Lemma 4.1. *Let $\phi(t)$ be a univariate function which can be expanded as a series of Legendre polynomials as in (1). If $\phi(t) \in C^{(2k+2j)}[-1,1]$, then*

$$|\mathcal{L}^k[\phi-\phi_L](x\cdot y)| \leqslant \frac{\mathcal{L}^{k+j}[\phi-\phi_L](1)}{(L+n-1)^{2j}} \leqslant \frac{\mathcal{L}^{k+j}\phi(1)}{(L+n-1)^{2j}}.$$

Proof. We have

$$|\mathcal{L}^k\phi(x\cdot y) - \mathcal{L}^k\phi_L(x\cdot y)| \leqslant \sum_{\ell \geqslant L+1} \lambda_\ell^k a_\ell |P_\ell(n+1; x\cdot y)|.$$

Since the Legendre polynomials satisfy the inequality $|P_\ell(n+1;t)| \leqslant P_\ell(n+1;1) = 1$ for every t in $[-1,1]$ (see [17, p. 15]), we have

$$\begin{aligned}\sum_{\ell>L}\lambda_\ell^k a_\ell P_\ell(n+1;t) &\leqslant \sum_{\ell>L}\lambda_\ell^k a_\ell P_\ell(n+1;1)\\ &\leqslant (L+n-1)^{-2j}\sum_{\ell>L}\lambda_\ell^{k+j} a_\ell P_\ell(n+1;1)\\ &\leqslant \frac{\mathcal{L}^{k+j}[\phi-\phi_L](1)}{(L+n-1)^{2j}}.\end{aligned}$$

The lemma follows by observing that $\mathcal{L}^{k+j}[\phi-\phi_L](1) \leqslant \mathcal{L}^{k+j}\phi(1)$. □

We are now in a position to obtain an error estimate for $\Delta^s(u-u_X)$, where u_X is the interpolant of u on the set X.

Proposition 4.2. *Suppose Φ is a positive definite kernel of the form (3), $\phi(t) \in C^{4s}[-1,1]$, and let X be a finite set of distinct points on S^n with mesh norm $h_X \leqslant 1/(2L)$. If u belongs to the native space N_Φ and u_X is an interpolant of the form (2) which interpolates u on the set X, then there is a constant $C > 0$ independent of u and X so that*

$$\|\Delta^s u - \Delta^s u_X\| \leqslant C \left(\sum_{\ell > L}^{\infty} \widehat{\phi}(\ell) N(n,\ell) \lambda_\ell^{2s} \right)^{1/2} \|u\|_\Phi.$$

Proof. Recalling the distributional framework set out in Section 2.3, we consider the following particular linear functional:

$$\eta(u) = \Delta^s u(x).$$

For a given point $x \in S^n$, we shall use inequality (7) to estimate $|\Delta^s u(x) - \Delta^s u_X(x)|$. Now Theorem 2.2 and Proposition 4.1 provide the following bound:

$$(P^\eta_{\Phi,W})^2 \leqslant \varDelta_0 + 4D_n^s L^{2s} \varDelta_1 + 4D_n^{2s} L^{4s} \varDelta_2,$$

where the Δ_j's are given by

$$\varDelta_0 = |\mathcal{L}^{2s}\phi(1) - \mathcal{L}^{2s}\phi_L(1)|,$$

$$\varDelta_1 = \max_j |\mathcal{L}^s \phi(x \cdot x_j) - \mathcal{L}^s \phi_L(x \cdot x_j)|,$$

and

$$\varDelta_2 = \max_{j,k} |\phi(x_k \cdot x_j) - \phi_L(x_k \cdot x_j)|.$$

Applying Lemma 4.1 to bound these quantities and then using the resulting bounds in the power-function estimate above, we obtain

$$\begin{aligned}(P^\eta_{\Phi,W})^2 &\leqslant \left(1 + \frac{4D_n^s L^{2s}}{(L+n-1)^{2s}} + \frac{4D_n^{2s} L^{4s}}{(L+n-1)^{4s}}\right) \mathcal{L}^{2s}[\phi - \phi_L](1) \\ &\leqslant C\mathcal{L}^{2s}[\phi - \phi_L](1),\end{aligned}$$

where C is a constant that depends only on n and s. The required result follows from the relation

$$\mathcal{L}^{2s}[\phi - \phi_L](1) = \frac{1}{|S^n|} \sum_{\ell > L} \lambda_\ell^{2s} \widehat{\phi}(\ell) N(n,\ell). \qquad \square$$

We now derive a simple consequence for our choice of kernels.

Corollary 4.2. *Suppose $\phi(t) \in C^{4s}[-1,1]$, and $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\tau}$ for some $\tau > n/2 + 2s$. Assume that the mesh norm h_X of the set X satisfies the condition $1/(2L+2) \leqslant h_X \leqslant 1/2L$. Then there exists a constant $C > 0$ independent of u and X so that*

$$\|\Delta^s u - \Delta^s u_X\| \leqslant C h_X^{\tau - n/2 - 2s} \|u\|_\Phi.$$

Proof. Since $(1+\lambda_\ell) \leqslant C\ell^2$ and $N(n,\ell) = \mathcal{O}(\ell^{n-1})$ we have

$$\sum_{\ell=L+1}^{\infty} \widehat{\phi}(\ell) N(n,\ell) \lambda_\ell^{2s} \leqslant C \int_L^{\infty} x^{n-1+4s-2\tau}\, dx \leqslant C L^{n+4s-2\tau}.$$

The result follows directly from Proposition 4.2 and the condition

$$1/(2L+2) \leqslant h_X \leqslant 1/2L. \qquad \square$$

4.2. Estimate in the supremum norm

We first state several results concerning approximation of functions on S^n by spherical harmonics in $\mathcal{V}_L$. These results, obtained by Pawelke [22,23], involve the notions of spherical mean and spherical modulus of continuity (see below). We shall use Pawelke's results later in the section.

Let $u \in C(S^n)$, $x \in S^n$, and $0 < h \leqslant \pi$. We define the spherical mean of u over the spherical cap of radius h centered at x as follows:

$$T_h u(x) := \frac{1}{|S^{n-1}|(\sin\ h)^{n-1}} \int_{x\cdot y=\cos h} u(y)\, d\sigma_x(y),$$

where $d\sigma_x$ is the volume element corresponding to $x \cdot y = \cos(h)$. The spherical modulus of continuity of u is defined to be

$$\omega(u,\varepsilon) := \sup_{0<h\leqslant\varepsilon} \|T_h u - u\|, \quad \varepsilon > 0.$$

Given $u \in C(S^n)$, we define the distance from u to the polynomial space $\mathcal{V}_L$ in the usual manner:

$$\operatorname{dist}(u,\mathcal{V}_L) := \inf_{P\in\mathcal{V}_L} \|u - P\|.$$

Theorem 4.2 (Pawelke [22,23]). *Suppose $u \in C^{2k}(S^n)$ and $L \in \mathbb{Z}^+$. There is a constant M, independent of both u and L, for which*

$$\operatorname{dist}(u,\mathcal{V}_L) \leqslant M\omega(u;1/L), \ \textit{and}\ \operatorname{dist}(u,\mathcal{V}_L) \leqslant M^k L^{-2k} \|\Delta^k u\|, \quad k \in \mathbb{Z}^+.$$

The remaining approximation theorems that we will use in the proof deal with the norms of iterates of Δ applied to the best and near-best approximants from $\mathcal{V}_L$.

Theorem 4.3 (Pawelke [22, Satz 4.4]). *Suppose $u \in C^{2k}(S^n)$, and let P_L be the best approximation to u from $\mathcal{V}_L$, i.e., $\|u - P_L\| = \operatorname{dist}(u,\mathcal{V}_L)$. Then there exists a constant C, independent of u and L, for which*

$$\|\Delta^k P_L\| \leqslant C \|\Delta^k u\|.$$

The preceding theorem has been extended in [20] to a class of near-best approximants from $\mathcal{V}_L$.

Theorem 4.4 (Narcowich and Ward [20, Corollary 2.5]). *Let $u \in C^{2k}(S^n)$ and let $Q_L \in \mathcal{V}_L$ for $L = 1, 2, \ldots$, be a sequence of polynomials satisfying $\|u - Q_L\| \leqslant K\,\mathrm{dist}(u, \mathcal{V}_L)$, with K independent of u and L. Then there is a constant C_1, independent of f and L, such that*

$$\|\Delta^k Q_L\| \leqslant C_1 \|\Delta^k u\|.$$

In the proof of the main result, we need to construct for every $u \in C(S^n)$, spherical harmonics that are both near-best approximants to u from $\mathcal{V}_L$ and also interpolate u on the point set X. This is precisely the content of the following theorem:

Theorem 4.5 (Narcowich and Ward [20, Theorem 3.1]). *Let $X \subset S^n$ be a finite set of distinct points with separation radius q_X and let $\beta > 1$. If we set $L = \lceil \frac{M(\beta+1)}{q_X(\beta-1)} \rceil$, with M as in Theorem 4.2, then for $u \in C(S^n)$ there exists a spherical harmonic $Q_L \in \mathcal{V}_L$ that interpolates u on X and also satisfies the estimate*

$$\|u - Q_L\| \leqslant (1 + \beta)\mathrm{dist}(u, \mathcal{V}_L).$$

Lemma 4.2. *Suppose $u \in C^{2s}(S^n)$, where s is a positive integer, and let P_L be the best approximation to u from $\mathcal{V}_L$, i.e., $\|u - P_L\| = \mathrm{dist}(u, \mathcal{V}_L)$. Then there is a constant C, independent of u and L, such that*

$$\|\Delta^s u - \Delta^s P_L\| \leqslant C\,\mathrm{dist}(\Delta^s u, \mathcal{V}_L).$$

Proof. We prove the lemma by induction on s. We consider the case $s = 1$. Note that if Q is a spherical harmonic of degree L, for $L > 0$, then so is ΔQ, because spherical harmonics are eigenfunctions of $\varDelta$. Therefore, the space of all spherical harmonics of degree $\leqslant L$ except constants, denoted by $\mathcal{V}_L \backslash V_0$, is isomorphic to $\varDelta(\mathcal{V}_L \backslash V_0)$. Let Q be a polynomial in $\mathcal{V}_L$ without constant term so that ΔQ is the best approximation to Δu. So

$$\|\Delta u - \Delta Q\| = \mathrm{dist}(\Delta u, \mathcal{V}_L).$$

Let $R \in \mathcal{V}_L$ be the best approximation to $u - Q$, so that

$$\|R - (u - Q)\| = \mathrm{dist}(u - Q, \mathcal{V}_L) = \mathrm{dist}(u, \mathcal{V}_L).$$

Since P_L is unique, we obtain $P_L = R + Q$. By the estimate in Theorem 4.3,

$$\|\Delta R\| \leqslant C\|\Delta u - \Delta Q\| = C\,\mathrm{dist}(\Delta u, \mathcal{V}_L).$$

Thus

$$\|\Delta u - \Delta P_L\| \leqslant \|\Delta u - \Delta Q\| + \|\Delta R\| \leqslant 2C\,\mathrm{dist}(\Delta u, \mathcal{V}_L).$$

Now let $s > 1$, and suppose that there is a constant C_0 so that

$$\|\Delta^{s-1} u - \Delta^{s-1} P_L\| \leqslant C_0\,\mathrm{dist}(\Delta^{s-1} u, \mathcal{V}_L).$$

Using the induction hypothesis for Δu and ΔQ, we have

$$\|\Delta^{s-1}\Delta u - \Delta^{s-1}\Delta Q\| = \|\Delta^s u - \Delta^s Q\| \leqslant C_0\,\mathrm{dist}(\Delta^s u, \mathcal{V}_L).$$

Using Theorem 4.3 once again, we have

$$\|\Delta^s R\| \leqslant C_1 \|\Delta^s u - \Delta^s Q\| \leqslant C_2 \,\mathrm{dist}(\Delta^s u, \mathcal{V}_L),$$

where $C_2 = C_1 C_0$. Thus

$$\|\Delta^s u - \Delta^s P_L\| \leqslant \|\Delta^s u - \Delta^s Q\| + \|\Delta^s R\| \leqslant C_3 \,\mathrm{dist}(\Delta^s u, \mathcal{V}_L),$$

with $C_3 = \max(C_0, C_2)$. □

We extend the result of the previous lemma to a broader class of near best approximants to u.

Lemma 4.3. *Suppose that $u \in C^{2k}(S^n)$ and Q_L is a near best approximation to u from $\mathcal{V}_L$ in the sense that there is a constant K, independent of L and u, so that*

$$\|u - Q_L\| \leqslant K \,\mathrm{dist}(u, \mathcal{V}_L).$$

Then there exists a constant C_2 so that for any integer $s \leqslant k$,

$$\|\Delta^s u - \Delta^s Q_L\| \leqslant C_2 L^{-2k+2s} \|\Delta^k u\|.$$

Proof. Let P_L be the best approximation to u from $\mathcal{V}_L$. The preceding lemma implies the estimate

$$\|\Delta^s u - \Delta^s P_L\| \leqslant C \,\mathrm{dist}(\Delta^s u, \mathcal{V}_L).$$

By the Markov–Bernstein inequality,

$$\begin{aligned}\|\Delta^s P_L - \Delta^s Q_L\| &\leqslant D_n^s L^{2s} \|P_L - Q_L\| \\ &\leqslant D_n^s L^{2s} (\|P_L - u\| + \|u - Q_L\|) \\ &\leqslant D_n^s L^{2s} (K+1) \,\mathrm{dist}(u, \mathcal{V}_L).\end{aligned}$$

Combining the two estimates above, we obtain

$$\begin{aligned}\|\Delta^s u - \Delta^s Q_L\| &\leqslant \|\Delta^s u - \Delta^s P_L\| + \|\Delta^s P_L - \Delta^s Q_L\| \\ &\leqslant C_1 \,\mathrm{dist}(\Delta^s u, \mathcal{V}_L) + D L^{2s} \,\mathrm{dist}(u, \mathcal{V}_L),\end{aligned}$$

where $D := D_n^s (K+1)$. Now by the second part of Theorem 4.2,

$$\mathrm{dist}(\Delta^s u, \mathcal{V}_L) \leqslant M_1 L^{-2k+2s} \|\Delta^k u\|$$

and

$$\mathrm{dist}(u, \mathcal{V}_L) \leqslant M_2 L^{-2k} \|\Delta^k u\|,$$

so the required result follows by setting $C_2 = \max\{C M_1, D M_2\}$. □

Now we adapt the proof in [20] to estimate $\|u - u_X\|$ for $u \in C^{2k}(S^n)$, which is in general a larger space of functions than the native space induced by the kernel Φ.

Theorem 4.6. *Let Φ be an SBF satisfying $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\tau}$ and suppose that $\tau > 2k \geqslant n/2 + 2s$. Let X be a quasi-uniform (i.e. the mesh ratio is bounded) discrete subset of S^n with* *mesh norm h_X. If $u \in C^{2k}(S^n)$ and $u_X \in V_X$ interpolates u on X as in (2), then for any nonnegative integer $s < k - n/4$, there exists a constant C independent of u and X such that*

$$\|\Delta^s u - \Delta^s u_X\| \leqslant C h_X^{2k-2s-n/2} \|u\|_{2k}.$$

Proof. Applying Theorem 4.5 with $\beta = 3$, we obtain a $P_L \in \mathcal{V}_L$ that interpolates u on X, where $L = \lceil 2M/q_X \rceil$, M is as in Theorem 4.2, and

$$\|u - P_L\| \leqslant 4\,\mathrm{dist}(u, \mathcal{V}_L).$$

Let P_X be the interpolant of P_L in the space V_X; then

$$\|\Delta^s u - \Delta^s u_X\| \leqslant \|\Delta^s u - \Delta^s P_L\| + \|\Delta^s P_L - \Delta^s P_X\| + \|\Delta^s (P_X - u_X)\|. \tag{12}$$

Since $P_X(x_j) = P_L(x_j) = u(x_j) = u_X(x_j)$ for all $x_j \in X$ and both P_X and u_X lie in the same finite-dimensional space V_X, we have $P_X \equiv u_X$ and the final term in the previous inequality vanishes. By Lemma 4.3, we have the estimate

$$\|\Delta^s u - \Delta^s P_L\| \leqslant C_0 L^{-2k+2s} \|u\|_{2k}.$$

Now the assumption on $\widehat{\phi}(\ell)$ guarantees that Corollary 4.2 is applicable and, since the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_{H^\tau}$ are equivalent, we can estimate the second term on the right hand side of (12) as follows:

$$\|\Delta^s P_L - \Delta^s P_X\| \leqslant C_1 h_X^{\tau-n/2-2s} \|P_L\|_{H^\tau}.$$

Using the definition of the Sobolev norm and the fact that P_L is a polynomial, we obtain

$$\|P_L\|_{H^\tau} \leqslant (1+\lambda_L)^{\tau/2-k} \|P_L\|_{H^{2k}} \leqslant 2^k |S^n|^{1/2} (1+\lambda_L)^{\tau/2-k} \|P_L\|_{2k}.$$

Since $\|P_L\| \leqslant 5\|u\|$ by assumption, Theorem 4.4 implies that $\|\Delta^k P_L\| \leqslant R\|\Delta^k u\|$, so that

$$\|P_L\|_{2k} \leqslant \max\{5, R\} \|u\|_{2k}.$$

So, if we set $C_2 = 2^k |S^n|^{1/2} \max\{5, R\}$ then

$$\|\Delta^s P_L - \Delta^s P_X\| \leqslant C_2 h_X^{\tau-n/2-2s} (1+\lambda_L)^{\tau/2-k} \|u\|_{2k}. \tag{13}$$

From (12), (13) and $\lambda_L = L(L+n-1) \leqslant C L^2$, we find that

$$\begin{aligned}
\|\Delta^s u - \Delta^s u_X\| &\leqslant (C_1 L^{2s-2k} + C_2 L^{\tau-2k} h_X^{\tau-n/2-2s}) \|u\|_{2k} \\
&\leqslant (C_1 L^{n/2+2s-2k} + C_2 L^{\tau-2k} h_X^{\tau-n/2-2s}) \|u\|_{2k} \\
&\leqslant [C_1 (h_X L)^{n/2+2s-2k} + C_2 (h_X L)^{\tau-2k}] h_X^{2k-n/2-2s} \|u\|_{2k}.
\end{aligned}$$

Using the fact that $L = \lceil 2M/q_X \rceil = \lceil 2M\rho_X/h_X \rceil$, we get

$$\|\Delta^s u - \Delta^s u_X\| \leqslant (C_3 \rho_X^{n/2+2s-2k} + C_4 \rho_X^{\tau-2k}) h_X^{2k-n/2-2s} \|u\|_{2k}.$$

Finally, since $\rho_X \geqslant 1$ and $\tau > 2s + n/2$, it follows that

$$\|\Delta^s u - \Delta^s u_X\| \leqslant C \rho_X^{\tau-2k} h_X^{2k-n/2-2s} \|u\|_{2k}. \qquad \square$$

5. The main theorem

Recall from Section 3 the following weak formulation of (10):

$$\left\langle -\Delta u + \omega^2 u, v \right\rangle = \langle f, v \rangle, \quad \forall v \in H^1, \tag{14}$$

and the fact that its solution u is being approximated by u_h, which, in turn, satisfies the following condition:

$$u_h \in V_X := \operatorname{span}\{\Phi(x_i, \cdot) : x_i \in X\}$$

such that

$$\left\langle -\Delta u_h + \omega^2 u_h, \chi \right\rangle = \langle f, \chi \rangle, \quad \forall \chi \in V_X. \tag{15}$$

Having assembled all the necessary ingredients, we are now ready to give our error bound for $u - u_h$:

Theorem 5.1. *Assume that the exact solution u of the weak formulation* (14) *belongs to $C^{2k}(S^n)$. The approximate solution u_h of the Ritz Galerkin approximation problem* (15) *is constructed from shifts of a kernel of the form* (3) *satisfying $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\tau}$ (for some $\tau > 2k$) and a quasi-uniform discrete set $X \subset S^n$ with mesh norm h_X. Then there is a positive constant C independent of u and h_X such that*

$$\|u - u_h\|_{H^1} \leqslant C h_X^{2k-n/2-1} \|u\|_{2k}.$$

Proof. By Theorem 4.6, we have a constant $C_1 > 0$ so that

$$\|\Delta u - \Delta u_X\|_2 \leqslant \sqrt{|S^n|}\,\|\Delta u - \Delta u_X\| \leqslant C_1 h_X^{2k-n/2-2} \|u\|_{2k}.$$

By Theorem 2.1 we also have

$$\|u - u_X\|_2 \leqslant \sqrt{|S^n|}\,\|u - u_X\| \leqslant C_2 h_X^{2k-n/2} \|u\|_{2k}.$$

So by Lemma 3.3, we conclude that

$$\|u - u_X\|_{H^1} \leqslant C_3 h_X^{2k-n/2} \sqrt{1 + h_X^{-2}}\,\|u\|_{2k}.$$

Now, using Cea's lemma (Lemma 3.2), we obtain the final estimate

$$\begin{aligned} \|u - u_h\|_{H^1} &\leqslant C\|u - u_X\|_{H^1} \leqslant C C_3 h_X^{2k-n/2} \sqrt{1 + h_X^{-2}}\,\|u\|_{2k} \\ &\leqslant C_4 h_X^{2k-n/2-1} \|u\|_{2k}. \end{aligned}$$

In the proof, C_k, for $k = 1, 2, 3, 4$, are generic constants independent of u and X. □

6. Implementation on S^2

Problems arising in satellite tracking and physical geodesy are still challenging because of the nature of the acquired data. If the data are localized, approximation problems can be solved through application of methods designed for two-dimensional Euclidean space (cf., e.g., [8] and references therein). However, problems involve essentially the entire surface of the sphere, or a sufficiently large part that modeling the data as arising in two space is no longer appropriate. In this section, we present only a test example to illustrate the Galerkin method on S^2 in which the approximate solution is constructed from various scattered sets X with different cardinality. It is assumed that the data is available globally with sufficient density and uniformity. In the implementation, there are two main issues to be addressed: the quadrature rule used in approximating the bilinear form $a(u, v)$ and the construction of spherical basis functions.

Since $\Phi(x, y)$ is a zonal function, we can reduce the surface integrals in the bilinear form $a(\Phi(x_i, \cdot), \Phi(x_j, \cdot))$ into one-dimensional series of Legendre polynomials as discussed in Section 6.1. For the surface integrals $\langle f, \Phi(x_i, \cdot)\rangle$'s, we have to derive a quadrature rule over the surface of the unit sphere as in Section 6.2.

6.1. *Inner product of two zonal functions*

Let $\phi(t)$ and $\psi(t)$, for $t \in [-1, 1]$, be two zonal functions on S^2. We can expand $\phi(t)$ and $\psi(t)$ in terms of series of Legendre polynomials

$$\phi(t) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(t), \quad \psi(t) = \sum_{\ell=0}^{\infty} b_\ell P_\ell(t),$$

where

$$a_\ell = \frac{\int_{-1}^{+1} \phi(t) P_\ell(t)\, dt}{\int_{-1}^{+1} [P_\ell(t)]^2\, dt} = \frac{2\ell+1}{2} \int_{-1}^{+1} \phi(t) P_\ell(t)\, dt \tag{16}$$

and

$$b_\ell = \frac{\int_{-1}^{+1} \psi(t) P_\ell(t)\, dt}{\int_{-1}^{+1} [P_\ell(t)]^2\, dt} = \frac{2\ell+1}{2} \int_{-1}^{+1} \psi(t) P_\ell(t)\, dt. \tag{17}$$

In the approximation of the bilinear form $a(u, v) = \left\langle -\Delta u + \omega^2 u, v \right\rangle$, we need the following useful lemma:

Lemma 6.1. *Let $\Psi(x, y) = \psi(x \cdot y)$ and $\Phi(x, y) = \phi(x \cdot y)$ be two zonal functions on S^2. For two distinct fixed points $p, q \in S^2$, the following relation holds*:

$$\int_{S^2} \phi(p \cdot x)\psi(q \cdot x)\, dS(x) = 4\pi \sum_{\ell=0}^{\infty} \frac{a_\ell b_\ell}{(2\ell+1)} P_\ell(p \cdot q).$$

Proof. We have

$$\phi(p \cdot x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(p \cdot x) = 4\pi \sum_{\ell=0}^{\infty} \frac{a_\ell}{(2\ell+1)} \sum_{k=-\ell}^{\ell} Y_{\ell,k}(p)\overline{Y_{\ell,k}(x)}$$

and

$$\psi(q\cdot x)=\sum_{\ell=0}^{\infty} b_\ell P_\ell(q\cdot x)=4\pi\sum_{\ell=0}^{\infty}\frac{b_\ell}{(2\ell+1)}\sum_{k=-\ell}^{\ell} Y_{\ell,k}(q)\overline{Y_{\ell,k}(x)}.$$

Since $\{Y_{\ell,k} : \ell = 0, 1, 2, \ldots ; k = -\ell \ldots \ell\}$ is an orthonormal set, we can use Parseval's identity to obtain

$$\begin{aligned}\int_{S^2}\phi(p\cdot x)\psi(q\cdot x)\,dS(x) &= 16\pi^2\sum_{\ell=0}^{\infty}\frac{a_\ell b_\ell}{(2\ell+1)^2}\sum_{k=-\ell}^{\ell} Y_{\ell,k}(p)\overline{Y_{\ell,k}(q)}\\ &= 16\pi^2\sum_{\ell=0}^{\infty}\frac{a_\ell b_\ell}{(2\ell+1)^2}\frac{(2\ell+1)}{4\pi}P_\ell(p\cdot q)\\ &= 4\pi\sum_{\ell=0}^{\infty}\frac{a_\ell b_\ell}{(2\ell+1)}P_\ell(p\cdot q). \qquad \square\end{aligned}$$

For numerical approximation, the integration in (16) and (17) can be approximated by a Gaussian quadrature formula over the interval $[-1, +1]$. If the function ϕ has $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\tau}$ as in condition (4) then $a_\ell \sim (1 + \lambda_\ell)^{-\tau}(2\ell + 1) \sim \ell^{-2\tau+1}$. We require that $\tau > 1/2$ and the one-dimensional Gaussian quadrature formula used in (16) and (17) should be exact up to polynomials of degree L. In our numerical experiments, $L = 6000$ and $\tau = 9/2$, so the smallest absolute value a_ℓ being computed is about 10^{-30}.

6.2. *Quadrature formula*

We seek a spherical quadrature rule that integrates exactly all polynomials up to a certain degree L, i.e., we seek a set of points $\Xi := \{\eta_1, \ldots, \eta_N\}$ and a set of positive weights $\{w_1, \ldots, w_N\}$ (Fig. 1) such that

$$\int_{S^2} P(x)\,dS=\sum_{j=1}^{N} w_j P(\eta_j), \qquad \forall P\in\mathcal{V}_L.$$

If all the weights are equal, namely $w_j = 4\pi/N$ for all $j = 1, \ldots, N$, then the set Ξ is called a spherical L-design, see [1,4,29]. It can be shown that a pair of antipodal points, the vertices of a regular tetrahedron, the regular octahedron, and the regular icosahedron give 1-, 2-, and 5-designs, respectively. If the weights are not equal and the points are chosen independent from the scattered data then there are many research directions which are still open [3,13,25,30].

The following existence theorem, proved in [14], provides in principle a possible quadrature formula for S^n.

Theorem 6.1. *Let L be an integer with $L \leqslant \alpha/h_\Xi$, where h_Ξ is the mesh norm of the set Ξ and α is some real constant. Then there exist nonnegative weights* $\{w_j : j = 1, \ldots, N\}$

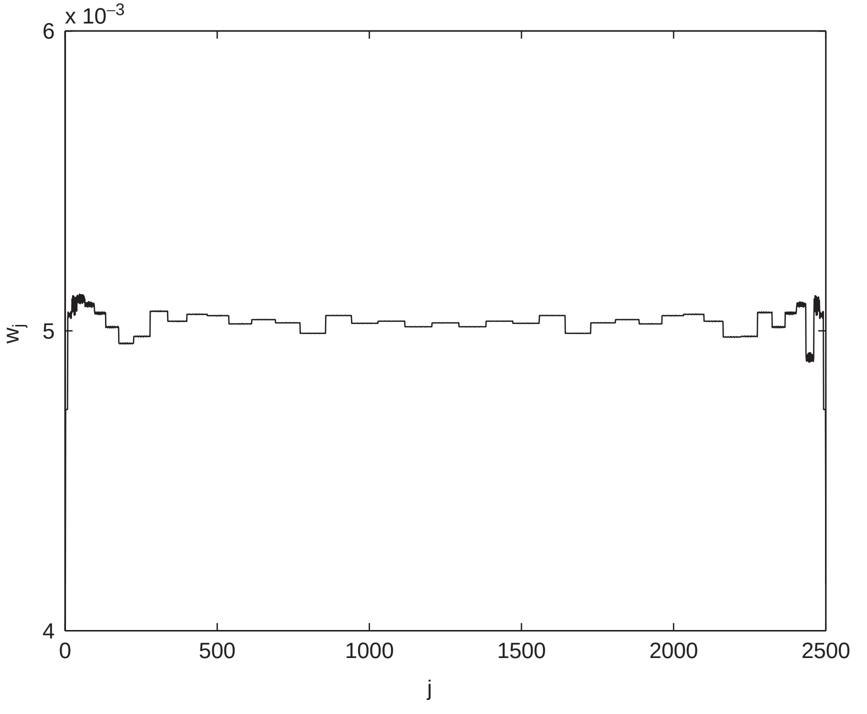

Fig. 1. Weights associated with 2500 quadrature points. The associated quadrature rule integrates exactly all polynomials up to degree 45.

such that

$$\int_{S^n} P(x)\,dS = \sum_{j=1}^{N} w_j P(\eta_j), \quad \forall P \in \mathcal{V}_L,$$

and the cardinality of the set of weights, N*, is comparable to the dimension of* $\mathcal{V}_L$.

Here we shall use the set of points that is constructed by dividing the surface of the sphere into N cells of roughly equal area (see [13]). Note that the set of quadrature points Ξ are constructed independently from the set of scattered data X. We investigate only the dependency between the rate of approximation and the mesh norm h_X of the scattered set X used to construct the approximate solution u_h.

Given $\mathbf{w} := \{w_1, \ldots, w_N\}$, the weights are computed by solving the following quadratic programming problem:

$$\min\ \mathbf{w} \cdot \mathbf{w}^T$$

subject to the following linear constraints:

$$\sum_{j=1}^{N} w_j Y_{\ell,k}(\eta_j) = 4\pi Y_{0,0}\delta_{\ell,0}, \quad \ell = 0, \ldots, L, \quad -\ell \leqslant k \leqslant \ell,$$
$$w_j \geqslant 0, \; j = 1, \ldots, N.$$

This optimization program can be solved numerically using the subroutine **quadprog** in MATLAB 6.0. The strategy is to start with a high value of L, say $L = \lfloor \sqrt{N} - 1 \rfloor$, and step it down by 1 until we reach a value of L for which we obtain a solution.

6.3. *The spherical basis functions*

In [32,33], Wendland introduced a class of locally supported positive definite radial basis functions defined on $\mathbb{R}^{n+1}$. These functions $\psi(x)$ are rotation invariant and are thus function of $|x|$ only. So the corresponding convolution kernel $\psi(x - y)$, $x, y \in S^n$, is a function of $|x - y| = \sqrt{2 - 2x \cdot y}$. We may therefore define a function

$$\Phi(x, y) = \phi(x \cdot y) := \psi(x - y), \quad x, y \in S^n. \tag{18}$$

Note that $\Phi(x, y)$ inherits the property of positive definiteness from ψ, and $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\tau}$ for some $\tau > 0$ (see Section 4 in [20]).

For our numerical study, we use the function $\psi(r) = (1 - r)_+^8 (32r^3 + 25r^2 + 8r + 1) \in C^6(\mathbb{R}^3)$, where $r = \sqrt{2 - 2x \cdot y}$. It is shown in [20] that the kernel $\Phi(x, y)$ induced by $\psi(r)$ satisfies $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{3/2+3} \sim \ell^9$. The support of $\psi(r)$ has radius 1, and hence for a fixed $x \in S^2$, the support of $\Phi(x, y)$ is $\{y \in S^2 : \cos\theta(x, y) \geqslant 1/2\}$, i.e., the spherical cap of radius $\pi/3$ centered at x. If we scale the support of ψ by a factor of $\alpha > 0$, the strictly positive definiteness of Φ is unchanged, but the rate of approximation will change according to α. The detailed results will be presented in a forthcoming paper.

Under the assumption that the collected data is abundant on the global scale, the sets of scattered data points X used in the construction of the SBFs are minimized energy points [30]. These points are generated using optimization packages, and are available at http://www.maths.unsw.edu.au/∼rsw/Sphere.

6.4. *Numerical results*

In our numerical experiments, we consider two examples: in one example the function f is a zonal function (i.e. $f(x)$ depends only on the geodesic distance from $x \in S^2$ to the north pole $(0, 0, 1)^T$) and in another, the function f is a spherical harmonic.

The quadrature rule used in the approximation of the bilinear form $a(\cdot, \cdot)$ and the surface integral $\langle f, \cdot \rangle$ are fixed, only the scattered set X for the SBFs varies in size.

The experiments use various values of ω, namely $\omega = 0.01; 0.1; 1.0; 10$. The errors are computed over a grid $\mathcal{C}$ of 10^4 points on the sphere. The ℓ_∞ errors are computed

Table 1
Errors for Example 1

ω	$m = \|X\|$	h_X(deg)	Rate	e_∞	Rate
0.01	64	17.5451		0.1324	
	225	9.1750	1.9123	0.0146	9.0684
	400	6.5092	1.4095	0.0026	5.6154
	784	5.3452	1.2177	9.3355e − 04	2.7851
	900	5.0092	1.0670	9.9907e − 04	0.9344
0.1	64	17.5451		0.1294	
	225	9.1750	1.9123	0.0078	16.5897
	400	6.5092	1.4095	0.0024	3.2500
	784	5.3452	1.2177	5.0605e − 04	4.7426
	900	5.0092	1.0670	5.4645e − 04	0.9261
1.0	64	17.5451		0.1105	
	225	9.1750	1.9123	0.0077	14.3506
	784	5.3452	1.7165	5.0050e − 04	15.3846
	900	5.0092	1.0671	7.3199e − 04	0.6837
	1681	3.6278	1.3808	9.0342e − 04	0.8102
10	64	17.5451		0.1193	
	225	9.1750	1.9123	0.0079	15.1013
	400	6.5092	1.4095	0.0024	3.2917
	784	5.3452	1.2177	6.8522e − 04	3.5025
	900	5.0092	1.0670	0.0020	

as follows:

$$e_\infty := \max_{\xi \in \mathcal{C}} |u(\xi) - u_h(\xi)|.$$

Example 1. We aim to solve numerically the following differential equation:

$$-\Delta u + \omega^2 u = -112(1 - \sqrt{2-2z})_+^4 (25z^2 - 9z + 4z\sqrt{2-2z} - 15),$$

where $(x, y, z) \in \mathbb{R}^3$ are points satisfying $x^2 + y^2 + z^2 = 1$. The exact solution of the differential equation is $u = (1 - \sqrt{2-2z})_+^6 (35(2-2z) + 18\sqrt{2-2z} + 3)$ which belongs to $C^4(S^2)$. In this example, since f is a zonal function, the integral $\langle f, \phi_j \rangle$ is approximated by a one-dimensional Gaussian rules used in computing $a(\cdot, \cdot)$ as mentioned in Section 6.1. The exact solution u belongs to $C^4(S^2)$, so Theorem 4.6 predicts the errors $\|u - u_h\|_{H^1}$ is about $\mathcal{O}(h_X^{4-2/2-1}) = \mathcal{O}(h_X^2)$. Table 1 shows that for ω close to 1.0 and q_X not too small (the condition number of the matrix A with $A_{ij} = [a(\phi_i, \phi_j)]$ is sensitive to the separation radius q_X of the discrete set X), the supremum errors $\|u - u_h\|$ can achieve up to $\mathcal{O}(h_X^4)$, which implies that the errors $\|u - u_h\|_{H^1}$ could be improved to $\mathcal{O}(h_X^3)$.

Example 2. We consider the approximation of the following differential equation:

$$-\Delta u + \omega^2 u = (2 + \omega^2) \sin(\phi) \cos(\theta), \qquad \phi \in [0, \pi], \ \theta \in [0, 2\pi].$$

Table 2
Errors for Example 2

ω	$m = \|X\|$	h_X (degree)	Rate	e_∞	Rate
0.01	784	5.3452		1.4326e − 05	
	900	5.0092	1.0671	2.6977e − 06	5.3104
	1600	3.7585	1.3328	1.0766e − 06	2.5058
	1681	3.6278	1.0360	1.1171e − 06	0.9637
	2500	2.9891	1.2137	1.9940e − 06	0.5602
0.1	784	5.3452		3.6315e − 06	
	900	5.0092	1.0671	3.4122e − 06	1.0642694
	1600	3.7585	1.3328	5.2482e − 07	6.5016577
	1681	3.6278	1.0360	5.2527e − 07	0.9991433
	2500	2.9891	1.2137	5.0058e − 07	1.0493228
1.0	784	5.3452		4.9329e − 06	
	900	5.0092	1.0671	3.0495e − 06	1.6176094
	1600	3.7585	1.3328	6.7267e − 07	4.5334265
	1681	3.6278	1.0360	6.0382e − 07	1.114024
	2500	2.9891	1.2137	4.6939e − 07	1.286393
10	784	5.3452		6.4366e − 06	
	900	5.0092	1.0671	3.9394e − 06	1.6339036
	1600	3.7585	1.3328	1.7974e − 06	2.1917214
	1681	3.6278	1.0360	1.6789e − 06	1.0705819
	2500	2.9891	1.2137	1.1070e − 06	1.5166215

The exact solution is $u(\phi, \theta) = \sin(\phi)\cos(\theta)$. In this example, the surface quadrature mentioned in Section 6.2 is used to approximate $\langle f, \phi_j \rangle$. The exact solution u is a spherical harmonic, so u belongs to the native space associated with the SBFs (cf. Section 6.2), which is $H^{9/2}(S^2)$, and hence by Corollary 4.2, the $\|u - u_h\|_{H^1}$ errors should be $\mathcal{O}(h_X^{9/2-2/2-1}) = \mathcal{O}(h_X^{5/2})$. Table 2 shows that under appropriate conditions for ω and q_X, the supremum errors $\|u - u_h\|$ can achieve up to $\mathcal{O}(h_X^4)$, which implies the errors $\|u - u_h\|_{H^1}$ could be improved to $\mathcal{O}(h_X^3)$.

Tables 1 and 2 show the errors between the exact solution and the approximate solution obtained via the Galerkin method using the SBFs $\Phi(x, y)$ centered at X as in (18).

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